$$= \bigcap_{i=1}^{\infty} [(A_i^1)^c \cup (A_i^2)^c \cup \ldots]$$
$$= [(A_1^1)^c \cup (A_1^2)^c \cup \ldots] \cap [(A_2^1)^c \cup (A_2^2)^c \cup \ldots]$$
$$\cap \ldots \cap [(A_n^1)^c \cup (A_n^2)^c \cup \ldots] \cap \ldots,$$

and this is equal to

$$\cup[(A_1^{i_1})^c (\cap A_2^{i_2})^c \cap \ldots \cap (A_n^{i_n})^c \cap \ldots]$$

with $i_1, i_2, \ldots, i_n, \ldots$ integers ≥ 1, and the union extends over all choices of the sets $(A_1^{i_1})^c, (A_2^{i_2})^c, \ldots, (A_n^{i_n})^c, \ldots$ from the respective collections: $(A_i^1)^c$, $(A_i^2)^c, \ldots, (A_i^n)^c, \ldots,\ i = 1, 2, \ldots, n, \ldots$ However, these choices produce $\aleph_0 \times \aleph_0 \times \ldots \times \aleph_0 \times \ldots = \aleph_0^{\aleph_0} = \aleph$ where $\aleph_0$ and $\aleph$ are the cardinal numbers of a countable set and of the continuum, respectively. Thus, there are uncountable members in the union, and hence the union need not be in $\mathcal{A}$. In other word, A^c need not be in $\mathcal{A}$, so that $\mathcal{A}$ need not be a σ-field.
Remark: For the justification of the equality, asserted in the derivations related to A^c, refer to the remark following the proof of Exercise 41. #

Chapter 2

Definition and Construction of a Measure and its Basic Properties

1. If Ω is finite, then μ is ≥ 0, $\mu(\oslash) = 0$ and finitely additive (since there are only finitely many subsets of Ω). Thus, μ is a measure, and also finite. If Ω is denumerable, $\Omega = \{\omega_1, \omega_2, \ldots\}$, then $\mu \geq 0$, $\mu(\oslash) = 0$, and if $A_n, n \geq 1$, are $\neq \oslash$ and pairwise disjoint, then $\mu\left(\sum_{n=1}^{\infty} A_n\right) = \infty$ and $\sum_{n=1}^{\infty} \mu\left(A_n\right) = \infty$ since each term is ≥ 1. Thus, μ is a measure. It is σ-finite, since $\Omega = \sum_{n=1}^{\infty}\{\omega_n\}$ and $\mu(\{\omega_n\}) = 1$ (finite). #

2. **(i)** Let $A_i \in \mathcal{C}, i = 1, \ldots, n, A_i \cap A_j = \oslash, i \neq j$, and set $A = \sum_{i=1}^{n} A_i$, so that $A \in \mathcal{C}$. Then either A is finite or A^c is finite. If A is finite, then all $A_i, i = 1, \ldots, n$, are finite, and therefore $P(A) = 0 = 0 + \ldots + 0 = P(A_1) + \ldots + P(A_n)$. If A^c is finite, then A is not finite and hence at least one of $A_1, \ldots, A_n$ is not finite; call A_{i_0} such an event. We claim that A_{i_0} is unique. Indeed, if A_i and $A_j, i \neq j$, are not finite, then A_i^c and A_j^c are finite. Since $A_i \cap A_j = \oslash$, it follows that $A_i \subset A_j^c$ and hence A_i is finite, a contradiction. Then $\sum_{i=1}^{n} P(A_i) = P(A_{i_0}) = 1$ (since $P(A_i) = 0, i \neq i_0$, as being all finite), and $P(A) = 1$. Hence $P(A) = \sum_{i=1}^{n} P(A_i)$.

(ii) Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ and take $A_i = \{\omega_i\}$, so that $A_i \cap A_j = \oslash$, $i \neq j$, and $P(A_i) = 0$ for all i. However, $P\left(\sum_{i=1}^{\infty} A_i\right)$ $(= P(\Omega)) = 1$ since $\sum_{i=1}^{\infty} A_i$ is infinite (and $\left(\sum_{i=1}^{n} A_i\right)^c = \oslash$ finite). Therefore $P\left(\sum_{i=1}^{\infty} A_i\right) = 1 \neq 0 = \sum_{i=1}^{\infty} P(A_i)$, and P is not σ-additive.

(iii) Let $A_n \in \mathcal{C}, n \geq 1, A_i \cap A_j = \oslash, i \neq j$, and set $A = \sum_{n=1}^{\infty} A_n$, so that $A \in \mathcal{C}$. Then either A is finite or A^c is finite. If A is finite, then all A_ns are finite (indeed, A is only the sum of finitely many of the A_ns) and hence $P(A_n) = 0$ for all n, and also $P(A) = 0$. Thus, $P(A) = \sum_{n=1}^{\infty} P(A_n)$ (actually, the σ-additivity here degenerates to finite additivity). If A^c is finite, then A is infinite. Since Ω is uncountable, it follows that at least one of the A_ns is infinite, because otherwise A would be countable (so that $A + A^c = \Omega$ is countable, a contradiction) ; call A_{n_0} such an event. We claim that A_{n_0} is unique. Indeed, if A_i and $A_j, i \neq j$, are infinite, then A_i^c and A_j^c are finite. Since $A_i \cap A_j = \oslash$, it follows that $A_i \subset A_j^c$ and hence A_i is finite, a contradiction. Then $\sum_{n=1}^{\infty} P(A_n) = P(A_{n_0}) = 1$ (since $P(A_n) = 0, n \neq n_0$, as being all finite), and $P(A) = 1$. Hence $P(A) = \sum_{n=1}^{\infty} P(A_n)$.
Finally, it is clear that $P(A) \geq 0$, $P(\oslash) = 0$ and $P(\Omega) = 1$. These properties along with the σ-additivity just established make P a probability measure. #

3. Clearly, $P(A) \geq 0, P(\oslash) = 0$ and $P(\Omega) = 1$ since $\Omega^c = \oslash$ countable. It remains to establish σ-additivity. Let $A_n \in \mathcal{C}, n \geq 1, A_i \cap A_j = \oslash, i \neq j$, and set $A = \cup_{n=1}^{\infty} A_n$. Since $A \in \mathcal{C}$, it follows that either A is countable or A^c is countable. If A is countable, then all A_ns are countable, and hence $P(A) = 0$ and $P(A_n) = 0, n \geq 1$, so that $P(A) = \sum_{n=1}^{\infty} P(A_n)$. If A^c is countable, then A is uncountable, and therefore at least one of the A_ns is uncountable; call A_{n_0} such an event. We claim that A_{n_0} is unique. Indeed, if A_i and $A_j, i \neq j$, are uncountable, then A_i^c and A_j^c are countable. Since $A_i \cap A_j = \oslash$, it follows that $A_i \subset A_j^c$ and hence A_i is countable, a contradiction. Then $\sum_{n=1}^{\infty} P(A_n) = P(A_{n_0}) = 1$ (since $P(A_i) = 0, i \neq n_0$, as being all countable), and $P(A) = 1$. Hence $P(A) = \sum_{n=1}^{\infty} P(A_n)$. #

4. $P(A_n) = 1$ if and only if $P(A_n^c) = 0$, which implies that $P\left(\cup_{n=1}^{\infty} A_n^c\right) \leq \sum_{n=1}^{\infty} P(A_n^c) = 0$; i.e., $P\left(\cup_{n=1}^{\infty} A_n^c\right) = 0$ or $P\left[\left(\cap_{n=1}^{\infty} A_n\right)^c\right] = 0$, and hence $P\left(\cap_{n=1}^{\infty} A_n\right) = 1$. #

5. For each $n \geq 2$, there are at most $n-1$ events A_is for which $P(A_i) > \frac{1}{n}$, because otherwise, we could choose n events with $P(A_{i_j}) > \frac{1}{n}$, so that $\sum_{j=1}^{n} P(A_{i_j}) > 1$. However, $\sum_{j=1}^{n} A_{i_j} \subseteq \Omega$ and $\sum_{j=1}^{n} P(A_{i_j}) = P\left(\sum_{j=1}^{n} A_{i_j}\right)$ (by pairwise disjointness), and this is $\leq P(\Omega) = 1$, a contradiction. Thus, if $I_n = \{i \in I;\, P(A_i) > \frac{1}{n}\}$, then the cardinality of I_n is $\leq n-1$. Set $I_0 = \{i \in I;\, P(A_i) > 0\}$. Then, clearly, $I_0 = \cup_{n=2}^{\infty} I_n$, and since each I_n is finite, I_0 is countable. #

6. Clearly, $\mu(A) \geq 0$ and $\mu(\oslash) = 0$. To establish σ-additivity. To this end, let $A_n \in \mathcal{A}$, $A_i \cap A_j = \oslash, i \neq j$, and set $A = \sum_{n=1}^{\infty} A_n$. Then:

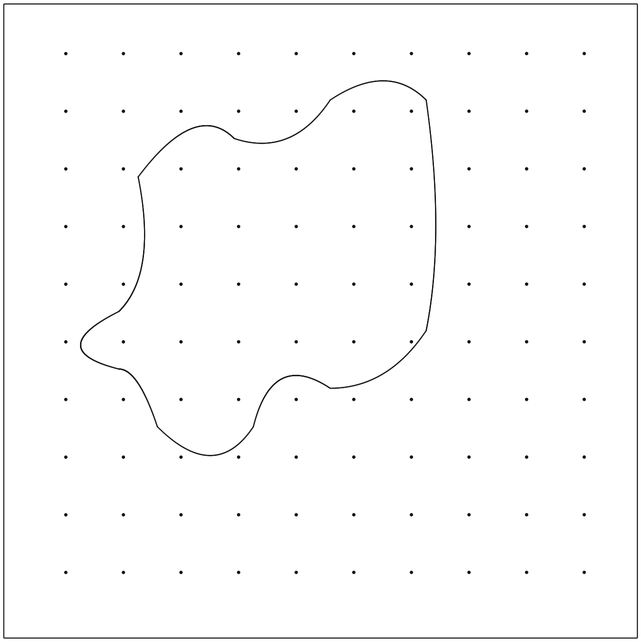
$$\mu(A) = \sum_{\omega_n \in A} p_n = \sum_{i=1}^{\infty} \sum_{\omega_n \in A_i} p_n = \sum_{i=1}^{\infty} \mu(A_i). \#$$

7. Let $\Omega_+ = \{\omega_n\text{s};\ p_n > 0\}$. Then the atoms are those A which are of the form: $A = \{\omega_n\} \cup N$, where $\oslash \subseteq N \subseteq \Omega - \Omega_+$. #

8. $\mu\left(\underline{\lim}_{n\to\infty} A_n\right) = \mu\left(\cup_{n=1}^{\infty}\cap_{i=n}^{\infty} A_i\right) = \mu\left(\lim_{n\to\infty}\cap_{i=n}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\cap_{i=n}^{\infty} A_i\right) \le \underline{\lim}_{n\to\infty}\, \mu(A_n)$ since $\cap_{i=n}^{\infty} A_i \subseteq A_n$. Next, $\mu\left(\overline{\lim}_{n\to\infty} A_n\right) = \mu\left(\cap_{n=1}^{\infty}\cup_{i=n}^{\infty} A_i\right) = \mu\left(\lim_{n\to\infty}\cup_{i=n}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\cup_{i=n}^{\infty} A_i\right)$, provided $\mu\left(\cup_{i=n}^{\infty} A_i\right) < \infty$ for some n, and this is $\ge \overline{\lim}_{n\to\infty}\mu(A_n)$ since $\cup_{i=n}^{\infty} A_i \supseteq A_n$. #

9. μ^0 is an outer measure; i.e., $\mu^0(\oslash) = 0$, μ^0 is $\uparrow$, and μ^0 is sub-σ-additive, because: $\mu^0(\oslash) = I_{\oslash}(\omega) = 0$; $A \subseteq B$ implies $I_A(\omega_0) \le I_B(\omega_0)$, so that $\mu^0(A) = I_A(\omega_0) \le I_B(\omega_0) = \mu^0(B)$; clearly, $I_{\cup_{i=1}^{\infty} A_i}(\omega_0) \le \sum_{i=1}^{\infty} I_{A_i}(\omega_0)$, so that $\mu^0\left(\cup_{i=1}^{\infty} A_i\right) = I_{\cup_{i=1}^{\infty} A_i}(\omega_0) \le \sum_{i=1}^{\infty} I_{A_i}(\omega_0) = \sum_{i=1}^{\infty} \mu^0(A_i)$. #

10.



That $\mu^0(\oslash) = 0$ and $\uparrow$ are obvious. Denote by C_i the i-th column, $i = 1, \ldots, 10$, and let $A_n \subseteq \Omega, n \ge 1$. To show $\mu^0\left(\cup_{n\ge1} A_n\right) \le \sum_{n\ge1} \mu^0(A_n)$. Set $A = \cup_{n\ge1} A_n$ and suppose $\mu(A) = k$. Then there exist k columns $C_{i_1}, \ldots, C_{i_k}$ such that $C_{i_j} \cap A \ne \oslash$, $j = 1, \ldots, k$. This implies that there exists at least one $x_j \in C_{i_j} \cap A$ with $x_j \in C_{i_j}$ and $x_j \in A$, so that $x_j \in C_{i_j}$ and $x_j \in A_{n_j}$, $j = 1, \ldots, k$, where $n_1, \ldots, n_k$ are chosen from the set $\{1, 2, \ldots\}$ and need not be distinct. Then $\mu^0(A_{n_j}) \ge 1$, $j = 1, \ldots, k$, and therefore:

$$k \le \sum_{j=1}^{k} \mu^0(A_{n_j}) \le \sum_{n\ge1} \mu^0(A_{n_j}) \text{ or } \mu^0\left(\underset{n\ge1}{\cup} A_n\right) \le \sum_{n\ge1} \mu^0(A_n). \#$$

11.

(i) In the first place, it is clear that $\mu^*(\oslash) = 0$ and $\mu^*(\Omega) = 1$. Next, let $\oslash \subset A \subset \Omega$. The only covering of A by member of $\mathcal{F}$ is Ω, so that $\mu^*(A) = 1$. Thus, $\mu^*(A) = 0$ if $A = \oslash$ and $\mu^*(A) = 1$ if $A \neq \oslash$.

(ii) First, $\oslash$ and Ω are μ^*-measurable, and let $\oslash \subset A \subset \Omega$ (which implies $\oslash \subset A^c \subset \Omega$). Then A cannot be μ^*-measurable. Indeed, in the required equality $\mu^*(D) = \mu^*(A \cap D) + \mu^*(A^c \cap D)$, take $D = \Omega$. Then the left-hand side is $\mu^*(D) = \mu^*(\Omega) = 1$, and the right-hand side is $\mu^*(A \cap \Omega) + \mu^*(A^c \cap \Omega) = \mu^*(A) + \mu^*(A^c) = 1 + 1 = 2$, and the equality is violated. Hence $\mathcal{A}^* = \{\oslash, \Omega\}$. #

12.

(i) $\mathcal{C}$ is not a field because, e.g., $\{\omega_1, \omega_2\} \cup \{\omega_1, \omega_3\} = \{\omega_1, \omega_2, \omega_3\} \notin \mathcal{C}$.

(ii) Clearly, $\mu(A) \geq 0$ and $\mu(\oslash) = 0$. The only two disjoint sets whose sum is also in $\mathcal{C}$ are: $\{\omega_1, \omega_2\} + \{\omega_3, \omega_4\} = \Omega$, $\{\omega_1, \omega_3\} + \{\omega_2, \omega_4\} = \Omega$, and, by taking measures, we have: $3+3 = 6$, $3+3 = 6$, so that μ is a measure.

(iii) On $\mathcal{C}$: $\mu_1(\oslash) = \mu_2(\oslash) = 0$, $\mu_1(\Omega) = \mu_2(\Omega) = 6$, $\mu_1(\{\omega_1, \omega_2\}) = 3 = \mu_2(\{\omega_1, \omega_2\})$, $\mu_1(\{\omega_1, \omega_3\}) = 3 = \mu_2(\{\omega_1, \omega_2\})$, $\mu_1(\{\omega_2, \omega_4\}) = 3 = \mu_2(\{\omega_2, \omega_4\})$, $\mu_1(\{\omega_3, \omega_4\}) = 3 = \mu_2(\{\omega_3, \omega_4\})$, so that $\mu_1 = \mu_2$ on $\mathcal{C}$.

(iv) Write out the subsets of Ω and their coverages by unions of members of $\mathcal{C}$ with the smallest measures to get:

$$
\begin{aligned}
\omega_1 &: \{\omega_1, \omega_2\} \\
\omega_2 &: \{\omega_1, \omega_2\} \\
\omega_3 &: \{\omega_1, \omega_3\} \\
\omega_4 &: \{\omega_2, \omega_4\} \\
\{\omega_1, \omega_2\} &: \{\omega_1, \omega_2\} \\
\{\omega_1, \omega_3\} &: \{\omega_1, \omega_3\} \\
\{\omega_1, \omega_4\} &: \{\omega_1, \omega_2\} \cup \{\omega_2, \omega_4\} \cup, \{\omega_1, \omega_2\} \\
&\quad \cup \{\omega_3, \omega_4\}, \{\omega_1, \omega_3\} \cup \{\omega_2, \omega_4\}, \\
&\quad \{\omega_1, \omega_3\} \cup \{\omega_3, \omega_4\} \\
\{\omega_2, \omega_3\} &: \{\omega_1, \omega_2\} \cup \{\omega_1, \omega_3\} \cup, \{\omega_1, \omega_2\} \\
&\quad \cup \{\omega_3, \omega_4\}, \{\omega_1, \omega_3\} \cup \{\omega_2, \omega_4\}, \\
&\quad \{\omega_2, \omega_4\} \cup \{\omega_3, \omega_4\} \\
\{\omega_2, \omega_4\} &: \{\omega_2, \omega_4\} \\
\{\omega_3, \omega_4\} &: \{\omega_3, \omega_4\} \\
\{\omega_1, \omega_2, \omega_3\} &: \{\omega_1, \omega_2\} \cup \{\omega_1, \omega_3\} \\
\{\omega_1, \omega_2, \omega_4\} &: \{\omega_1, \omega_2\} \cup \{\omega_2, \omega_4\} \\
\{\omega_1, \omega_3, \omega_4\} &: \{\omega_1, \omega_3\} \cup \{\omega_2, \omega_4\} \\
\{\omega_2, \omega_3, \omega_4\} &: \{\omega_2, \omega_4\} \cup \{\omega_3, \omega_4\}. \text{ Then:}
\end{aligned}
$$

$$\mu^*(\{\omega_1\}) = \mu^*(\{\omega_2\}) = \mu^*(\{\omega_3\}) = \mu^*(\{\omega_4\}) = 3,$$
$$\mu^*(\{\omega_1,\omega_2\}) = \mu^*(\{\omega_1,\omega_3\}) = \mu^*(\{\omega_2,\omega_4\}) = \mu^*(\{\omega_3,\omega_4\}) = 3,$$
$$\mu^*(\{\omega_1,\omega_4\}) = \mu^*(\{\omega_2,\omega_3\}) = 6,$$
$$\mu^*(\{\omega_1,\omega_2,\omega_3\}) = \mu^*(\{\omega_1,\omega_2,\omega_4\}) = \mu^*(\{\omega_1,\omega_3,\omega_4\}) = \mu^*(\{\omega_2,\omega_3,\omega_4\}) = 6.$$

(v) By part (iv), $\mu^* \neq \mu_1 \neq \mu_2$ because, e.g., $\mu_1(\{\omega_1,\omega_4\}) = 2$, $\mu_2(\{\omega_1,\omega_4\}) = 4$ and $\mu^*(\{\omega_1,\omega_4\}) = 6$, all distinct. #

13. **(i)** Immediate.

(ii) The only partition of Ω with members in $\mathcal{C}$ is $\{A, A^c\}$ and $\mu(A) = \mu(A^c) = 0$ ($A^c = \{0, 2, 4, \ldots\}$).

(iii) On $\mathcal{C}$, $\mu_1(\oslash) = \mu_2(\oslash) = 0$, and $\mu_1(A) = \mu_1(A^c) = \mu_1(\Omega) = \infty = \mu_2(A) = \mu_2(A^c) = \mu_2(\Omega)$.

(iv) Let $\oslash \subset B \subset \Omega$. Then the only possible coverages of B by members of $\mathcal{C}$ are: A, A^c, Ω, all of which have μ-measure ∞. Thus, $\mu^*(B) = \infty$ for every B as above.

(v) Let $\oslash \subset B \subset \Omega$. Then if $D \subset \Omega$ is $= \oslash$, from $\oslash = (B \cap \oslash) + (B^c \cap \oslash)$, it follows that $0 = 0$, whereas for $D \neq \oslash$, the relation $D = (B \cap D) + (B^c \cap D)$ implies that at least one of $B \cap D$ and $B^c \cap D$ is $\neq \oslash$. Hence $\infty = \infty$ and the equality holds again. Since $\oslash$ and Ω are always μ^*-measurable, it follows that $\mathcal{A}^* = \mathcal{P}(\Omega)$. #

15. **(i)** To show that $A \triangle M = (A - N) \cup [N \cap (A \triangle M)]$, where $M \subseteq N$. We have

$$\begin{aligned} A \triangle M &= (A \triangle M) \cap \Omega = (A \triangle M) \cap (N \cup N^c) \\ &= [(A \triangle M) \cap N] \cup [(A \triangle M) \cap N^c] \\ &= [N \cap (A \triangle M)] \cup \{[(A - M) \cup (M - A)] \cap N^c\} \\ &= [N \cap (A \triangle M)] \cup \{[(A \cap M^c) \cup (A^c \cap M)] \cap N^c\} \\ &= [N \cap (A \triangle M)] \cup (A \cap M^c \cap N^c) \cup (A^c \cap M \cap N^c) \\ &= [N \cap (A \triangle M)] \cup (A \cap M^c \cap N^c) \\ &\quad (\text{since } M \subseteq N \text{ implies } N^c \subseteq M^c \text{ and hence } M \cap N^c = \oslash) \\ &= [N \cap (A \triangle M)] \cup (A \cap N^c) \ (\text{since } N^c \subseteq M^c) \\ &= (A - N) \cup [N \cap (A \triangle M)]. \end{aligned}$$

(ii)

$$\begin{aligned} A \cup M &= [(A - N) \cup (A \cap N)] \cup M \\ &= (A - N) \cup [(A \cap N) \cup M] \\ &= (A - N) \cup [(A \cap N) \cup (M \cap N)] \ (\text{since } M \subseteq N) \\ &= (A - N) + [(A \cup M) \cap N] \\ &= (A - N) \triangle [N \cap (A \cup M)] \\ &\quad (\text{since for } B \text{ and } C \text{ with } B \cap C = \oslash,\ B + C = B \triangle C). \end{aligned}$$

(iii) Let $B \in \mathcal{A}^*$. Then $B = A \triangle M$ for some $A \in \mathcal{A}$ and $M \subseteq N$, $N \in \mathcal{A}$ with $\mu(N) = 0$. By part (i), $B = A \triangle M = (A - N) \cup [N \cap (A \triangle M)]$ with $(A - N) \in \mathcal{A}$ and $N \cap (A \triangle M) \subseteq N$. That is, B is of the form $A \cup M$ with A replaced by $A - N$ (a member of $\mathcal{A}$) and M replaced by $N \cap (A \triangle M)$ (which is a subset of N with $N \in \mathcal{A}$ and $\mu(N) = 0$). It follows that $B \in \bar{\mathcal{A}}$. Next, let $B \in \bar{\mathcal{A}}$. Then $B = A \cup M$ for some $A \in \mathcal{A}$ and some $M \subseteq N$ with $N \in \mathcal{A}$ and $\mu(N) = 0$. By part (ii), $B = A \cup M = (A - N) \triangle [N \cap (A \cup M)]$ with $(A - N) \in \mathcal{A}$ and $N \cap (A \cup M) \subseteq N$. That is, B is of the form $A \cup M$ with A replaced by $A - N$ (a member of $\mathcal{A}$) and M replaced by $N \cap (A \cup M)$ (which is a subset of $N \in \mathcal{A}$ and $\mu(N) = 0$). It follows that $B \in \mathcal{A}^*$. Therefore $\mathcal{A}^* = \bar{\mathcal{A}}$.

Note: Parts (i) and (ii) are also established by showing that each side is contained in the other. This is done as follows.

(i) Let ω belong to the left-hand side; i.e., $\omega \in A \triangle M$, so that $\omega \in A$ and $\omega \notin M$. That $\omega \notin M$ implies that either $\omega \notin N$ or $\omega \in N$. If $\omega \notin N$, then $\omega \in (A - N)$, so that ω belongs to the right-hand side. If $\omega \in N$, then $\omega \in [N \cap (A \triangle M)]$, so that ω belongs to the right-hand side again.
Next, let ω belong to the right-hand side. Then $\omega \in (A - N)$ or $\omega \in [N \cap (A \triangle M)]$. If $\omega \in (A - N)$, then $\omega \in A$ and $\omega \notin N$, so that $\omega \in A$ and $\omega \notin M$. It follows that ω belongs to the left-hand side. On the other hand, if $\omega \in [N \cap (A \triangle M)]$, then $\omega \in (A \triangle M)$, so that ω belongs to the left-hand side again.

(ii) Let ω belong to the left-hand side; i.e., $\omega \in A \cup M$, so that $\omega \in A$ and $\omega \in M$ or to both. Let $\omega \in A$. Also, either $\omega \in N$ or $\omega \notin N$. If $\omega \in N$, then $\omega \in [N \cap (A \cup M)]$, so that ω belongs to the right-hand side. If $\omega \notin N$, then $\omega \in (A - N)$, so that ω belongs to the right-hand side again. Finally, let $\omega \in M$. Then $\omega \in N$ and hence $\omega \in [N \cap (A \cup M)]$, so that ω belongs to the right-hand side.
Next, let ω belong to the right-hand side. Then either $\omega \in (A - N)$ or $\omega \in [N \cap (A \cup M)]$. Let $\omega \in (A - N)$. Then $\omega \in A$ and $(\omega \notin N)$, so that ω belongs to the left-hand side. If $\omega \in [N \cap (A \cup M)]$, then $\omega \in (A \cup M)$, so that ω belongs to the left-hand side. #

16. $\bar{\mathcal{A}}(= \mathcal{A}^*) \neq \oslash$ since, e.g., $\Omega = \Omega \cup \oslash$, $\Omega \in \mathcal{A}$, $\mu(\oslash) = 0$, so that $\Omega \in \bar{\mathcal{A}}$. Next, for $B \in \bar{\mathcal{A}}$ to show that $B^c \in \bar{\mathcal{A}}$. Now $B \in \bar{\mathcal{A}}$ implies $B = A \cup M$, $A \in \mathcal{A}$, $M \subseteq N \in \mathcal{A}$, $\mu(N) = 0$. Then

$$\begin{aligned}
B^c &= (A \cup M)^c = A^c \cap M^c \\
&= A^c \cap [M^c \cap (N \cup N^c)] \\
&= A^c \cap [(M^c \cap N) \cup (M^c \cap N^c)] \\
&= A^c \cap [(M^c \cap N) \cup N^c] \text{ (since } M \subseteq N \text{ implies } N^c \subseteq M^c) \\
&= (A^c \cap N^c) \cup (N \cap M^c \cap A^c)
\end{aligned}$$

with $A^c \cap N^c \in \mathcal{A}$ and $N \cap M^c \cap A^c \subseteq N$. That is, B^c is of the form $A \cup M$ with A (a member of $\mathcal{A}$) replaced by $A^c \cap N^c$ and M ($\subseteq N \in \mathcal{A}$ with $\mu(N) = 0$) replaced by $N \cap M^c \cap A^c$. It follows that $B^c \in \bar{\mathcal{A}}$. Finally, let $B_i \in \bar{\mathcal{A}}$, $i = 1, 2, \ldots$ Then $B_i = A_i \cup M_i$ with $A_i \in \mathcal{A}$ and $M_i \subseteq N_i \in \mathcal{A}$ with $\mu(N_i) = 0$, $i \geq 1$. Therefore

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A_i \cup M_i) = (\bigcup_{i=1}^{\infty} A_i) \cup (\bigcup_{i=1}^{\infty} M_i) \text{ with } \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

and $\cup_{i=1}^{\infty} M_i \subseteq \cup_{i=1}^{\infty} N_i$, a member of $\mathcal{A}$ with $\mu(\cup_{i=1}^{\infty} N_i) = 0$. It follows that $\cup_{i=1}^{\infty} B_i$ belongs in $\bar{\mathcal{A}}$, and $\bar{\mathcal{A}}$ is a σ-field. #

17.

(i) In the first place, the definition $\mu^*(A \triangle M) = \mu(A)$ implies $\mu^*(A \cup M) = \mu(A)$. Indeed, $A \cup M = (A - N) \triangle [N \cap (A \cup M)]$ with $(A - N) \in \mathcal{A}$ and $N \cap (A \cup M) \subseteq N \in \mathcal{A}$, $\mu(N) = 0$. Therefore $\mu^*(A \cup M) = \mu(A - N) = \mu(A \cap N^c) = \mu(A \cap N^c) + \mu(A \cap N) = \mu[(A \cap N^c) \cup (A \cap N)] = \mu(A)$. In the process of the proof, we also have seen that $\mu(A - N) = \mu(A)$.

(ii) As it was just seen, $\mu^*(A \cup M) = \mu(A - N) = \mu(A)$. We show that μ^* so defined on $\mathcal{A}^*$ is well-defined. That is, if $B = A_1 \cup M_1 = A_2 \cup M_2$, then $\mu(A_1) = \mu(A_2)$. Indeed,

$$A_1 = (A_1 \cap A_2) + (A_1 \cap A_2^c) = (A_1 \cap A_2) \triangle (A_1 \cap A_2^c).$$

Next, $A_1 \cap A_2^c \subseteq M_2$, because $x \in (A_1 \cap A_2^c)$ implies $x \in A_1$ and $x \notin A_2$, hence $x \in (A_1 \cup M_1)$ and $x \notin A_2$, so that $x \in B$ and $x \notin A_2$. This implies that $x \in (A_2 \cup M_2)$ and $x \notin A_2$, so that $x \in M_2$. Thus, $A_1 \cap A_2^c \subseteq M_2 \subseteq N_2$. From this and the fact that $B = (A_1 \cap A_2) \triangle (A_1 \cap A_2^c)$, it follows that $\mu^*(B) = \mu(A_1 \cap A_2)$ ($= \mu(A_1)$). Likewise, $A_2 = (A_1 \cap A_2) \triangle (A_1^c \cap A_2)$ with $A_1^c \cap A_2 \subseteq M_1 \subseteq N_1$, so that $\mu^*(B) = \mu(A_1 \cap A_2)$ ($= \mu(A_2)$). It follows that $\mu(A_1) = \mu(A_2)$ and μ^* is well-defined.

(iii) Clearly, $\mu^*(\emptyset) = \mu^*(\emptyset \triangle \emptyset) = \mu(\emptyset) = 0$, and $\mu^*(A \cup M) = \mu(A)$ (as was seen in part (i)) and this is ≥ 0. Finally, let $B_i \in \bar{\mathcal{A}}$, $i = 1, 2, \ldots$, $B_i \cap B_j = \emptyset$, $i \neq j$. Then $B_i = A_i \cup M_i$, $B_j = A_j \cup M_j$, and

$$\emptyset = B_i \cap B_j = (A_i \cap A_j) \cup (A_i \cap M_j) \cup (M_i \cap A_j) \cup (M_i \cap M_j),$$

so that $A_i \cap A_j = \emptyset$. Therefore

$$\mu^*\left(\sum_{i=1}^{\infty} B_i\right) = \mu^*\left[\bigcup_{i=1}^{\infty} (A_i \cup M_i)\right] = \mu^*\left[\left(\bigcup_{i=1}^{\infty} A_i\right) \cup \left(\bigcup_{i=1}^{\infty} M_i\right)\right]$$
$$= \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right)$$

$$= \sum_{i=1}^{\infty} \mu^*(A_i \cup M_i)$$
$$= \sum_{i=1}^{\infty} \mu^*\left(B_i\right).$$

It follows that μ^* is a measure on $\bar{\mathcal{A}}(=\mathcal{A}^*)$. #

18. **(i)** Let $B \in \hat{\mathcal{C}}$ and suppose that $B = A$ for some $A \in \mathcal{A}$. Then $B = A\Delta\oslash$ with $\oslash \in \mathcal{A}$ and $\mu(\oslash) = 0$, so that $B \in \mathcal{A}^*$. If $B \subseteq N$ for some $N \in \mathcal{A}$ with $\mu(N) = 0$, we have $B = B\Delta\oslash$ with $\oslash \in \mathcal{A}$ and $B \subseteq N \in \mathcal{A}$ with $\mu(N) = 0$, so that $B \in \mathcal{A}^*$. Thus $\hat{\mathcal{C}} \subseteq \mathcal{A}^*$.
(ii) $\hat{\mathcal{C}} \subseteq \mathcal{A}^*$ implies that $\sigma(\hat{\mathcal{C}}) = \hat{\mathcal{A}} \subseteq \mathcal{A}^*$, so it suffices to show that $\mathcal{A}^* \subseteq \hat{\mathcal{A}}$. Let $B \in \mathcal{A}^*$, so that $B = A\Delta M$ with $A \in \mathcal{A}$ and $M \subseteq N, N \in \mathcal{A}$, $\mu(N) = 0$. Since $A = A\Delta\oslash$, it follows that $A \in \hat{\mathcal{C}}$ and hence $A \in \hat{\mathcal{A}}$. Also, $M = \oslash\Delta M$, so that $M \in \hat{\mathcal{C}}$ and hence $M \in \hat{\mathcal{A}}$. Thus, $A, M \in \hat{\mathcal{A}}$ and therefore $A\Delta M \in \hat{\mathcal{A}}$ or $B \in \hat{\mathcal{A}}$. #

19. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and let $\mathcal{A} = \{\oslash, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$. Then $\mathcal{A}$ is, trivially, a σ-field. On $\mathcal{A}$, define μ as follows: $\mu(\oslash) = 0 = \mu(\{\omega_1, \omega_2\})$, $\mu(\{\omega_3, \omega_4\}) = \mu(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 1$. Then, clearly, μ is a measure on $\mathcal{A}$. But $\{\omega_1\} \subset \{\omega_1, \omega_2\} \in \mathcal{A}$ with $\mu(\{\omega_1, \omega_2\}) = 0$ whereas $\{\omega_1\} \notin \mathcal{A}$. #

20. Recall that μ^0 is an outer measure on $\mathcal{P}(\Omega)$ if $\mu^0(\oslash) = 0$, μ^0 is $\uparrow$ and sub-σ-additive. Now, let $N \in \mathcal{A}^0$ with $\mu^0(N) = 0$, and let M be an arbitrary subset of N. To show that $M \in \mathcal{A}^0$. It suffices to show that $\mu^0(D) \geq \mu^0(M \cap D) + \mu^0(M^c \cap D)$ for every $D \subseteq \Omega$. We have: $M \subseteq N$, hence $M \cap D \subseteq N \cap D$ and $\mu^0(M \cap D) \leq \mu^0(N \cap D) = 0$, so that $\mu^0(M \cap D) = 0$. Next, $M^c \cap D \subseteq D$ and $\mu^0(M^c \cap D) \leq \mu^0(D)$, so that $\mu^0(D) \geq \mu^0(M \cap D) + \mu^0(M^c \cap D)$ for every $D \subseteq \Omega$. #

21. On $\mathcal{B}$, define μ in the following manner: $\mu(B) =$ number of integers in B. Then, clearly, μ is a measure satisfying the condition $\mu(\text{finite interval}) < \infty$. Next, let $x_n \uparrow -2$, so that $\mu((x_n, 0]) = 3$ for all sufficiently large n, and hence $F_c(x_n) = c - 3$ for all sufficiently large n. But $F_c(-2) = c - \mu((-2, 0]) = c - 2$. Hence F_c is not left-continuous. #

22. Indeed, if μ were additive, then $c = \mu(\oslash) = \mu(\oslash \cup \oslash) = \mu(\oslash) + \mu(\oslash) = 2c$, so that $2 = 1$, a contradiction. #

23. For $n = 2$, let μ_1 and μ_2 be σ-finite, and let $\{A_1^1, A_2^1, \ldots\}$ and $\{A_1^2, A_2^2, \ldots\}$ be the associated partitions for which $\mu_1(A_i^1) < \infty$, $\mu_2(A_i^2) < \infty, i \geq 1$. Then $\{A_i^1 \cap A_j^2,\ i, j \geq 1\}$ is a partition of Ω and $\mu(A_i^1 \cap A_j^2) = \mu_1(A_i^1 \cap A_j^2) + \mu_2(A_i^1 \cap A_j^2) < \infty, i, j \geq 1$, so that μ is σ-finite.

Next, assume the assertion to be true for $n = k$ and we will establish it for $n = k + 1$. By setting $\mu_0 = \mu_1 + \ldots + \mu_k$, we have that both μ_0 and μ_{k+1} are σ-finite, and let $\{B_i,\ i \geq 1\}$ and $\{A_i^{k+1},\ i \geq 1\}$ be the associated partitions for which $\mu_0(B_i) < \infty$, $\mu_{k+1}(A_i^{k+1}) < \infty, i \geq 1$. Then $\{B_i \cap A_j^{k+1},\ i, j \geq 1\}$

is a partition of Ω, and $\mu_0(B_i \cap A_j^{k+1}) \leq \mu_0(B_i) < \infty$, $\mu_{k+1}(B_i \cap A_j^{k+1}) \leq \mu_{k+1}(A_j^{k+1}) < \infty$, $i, j \geq 1$. Thus,

$$(\mu_1 + \ldots + \mu_{k+1})(B_i \cap A_j^{k+1}) = (\mu_1 + \ldots + \mu_k)(B_i \cap A_j^{k+1}) + \mu_{k+1}(B_i \cap A_j^{k+1}) < \infty, \ i, j \geq 1, \text{ so that}$$

$\mu_1 + \ldots + \mu_{k+1}$ is σ-finite. #

24. **(i)** Clearly, $(A \cap B^c) \cup (A^c \cap B) = A \triangle B = (A \cup B) - (A \cap B)$. Hence

$$\begin{aligned} P[(A \cap B^c) \cup (A^c \cap B)] &= P[(A \cup B) - (A \cap B)] \\ &= P(A \cup B) - P(A \cap B) \text{ (since } A \cap B \subseteq A \cup B) \\ &= P(A) + P(B) - P(A \cap B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

(ii) We will use the induction hypothesis.
For $n = 2$, we have:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2),$$

so that

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1) + P(A_2) - P(A_1 \cup A_2) \\ &\geq P(A_1) + P(A_2) - 1. \end{aligned}$$

Next, assume it to be true for $n = k$ and establish it for $n = k + 1$. Indeed,

$$\begin{aligned} P(A_1 \cap \ldots \cap A_{k+1}) &= P[(A_1 \cap \ldots \cap A_k) \cap A_{k+1}] \\ &\geq P(A_1 \cap \ldots \cap A_k) + P(A_{k+1}) - 1 \\ &\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1 \\ &= \sum_{i=1}^{k+1} P(A_i) - [(k+1) - 1]. \ \# \end{aligned}$$

25. $\underline{\lim}_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \cup_{n=1}^{\infty}\{\omega_2\} = \{\omega_2\}$, $\overline{\lim}_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \cap_{n=1}^{\infty}\{\omega_1, \omega_2, \omega_3\} = \{\omega_1, \omega_2, \omega_3\}$, so that $P(\underline{\lim}_{n\to\infty} A_n) = P(\{\omega_2\}) = \frac{1}{3}$, $P(\overline{\lim}_{n\to\infty} A_n) = P(\{\omega_1, \omega_2, \omega_3\}) = \frac{7}{10}$; also, $P(A_{2n-1}) = P(\{\omega_1, \omega_2\}) = \frac{1}{2}$, $P(A_{2n}) = P(\{\omega_2, \omega_3\}) = \frac{8}{5}$, so that $\underline{\lim}_{n\to\infty} P(A_n) = \frac{1}{2}$ and $\overline{\lim}_{n\to\infty} P(A_n) = \frac{8}{5}$. Observe that

$$P(\underline{\lim}_{n\to\infty} A_n) = \frac{1}{3} \neq \frac{1}{2} = \underline{\lim}_{n\to\infty} P(A_n),$$

and

$$P(\overline{\lim}_{n\to\infty} A_n) = \frac{7}{10} \neq \frac{8}{5} = \overline{\lim}_{n\to\infty} P(A_n). \ \#$$

26. **(i)** If $\{\omega_i\} \in \mathcal{A}$ for all ω_i, then, clearly, every subset of Ω is in $\mathcal{A}$, so that $\mathcal{A} = \mathcal{P}(\Omega)$. On the other hand, if $\mathcal{A} = \mathcal{P}(\Omega)$, then all subjects of Ω are in $\mathcal{A}$, and in particular, so are $\{\omega_i\}$ for all ω_is.

(ii) It is immediate. #

27. **(i)** That $\mu(A) \geq 0$ and $\mu(\oslash) = 0$ are immediate. Next, let $A_1, \ldots, A_n$ be pairwise disjoint. Then to show that $\mu(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$. If at least one of the A_is is infinite, then $\sum_{i=1}^{n} A_i$ is infinite, so that $\mu(\sum_{i=1}^{n} A_i) = \infty$. Also, at least one of the terms on the right-hand side is ∞, so that $\sum_{i=1}^{n} \mu(A_i) = \infty$. On the other hand, if all $A_1, \ldots, A_n$ are finite, then $\sum_{i=1}^{n} A_i$ is finite and hence $\mu(\sum_{i=1}^{n} A_i) = 0$. The right-hand side is also equal to 0 since each term is 0. Next, μ is not σ-additive, because if all A_is are finite, then $\sum_{i=1}^{\infty} A_i$ is infinite, so that $\mu(\sum_{i=1}^{\infty} A_i) = \infty$, whereas $\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} 0 = 0$.

(ii) Clearly, $\Omega = \cup_{n=1}^{\infty} A_n$, where $A_n = \{\omega_1, \ldots, \omega_n\}$, so that $A_n \subset A_{n+1}$, $n \geq 1$, and $\mu(A_n) = 0$ for all n. Since $\mu(A_n) = 0, n \geq 1$, it follows that $\mu(A_n^c) = \infty$ for all n. #

28. **(i)** We have to prove that $\mu^0(\oslash) = 0$, $\mu^0(A) \leq \mu^0(B)$ for $A \subset B$, and μ^0 is a sub-σ-additive. That $\mu^0(\oslash) = 0$ holds by the definition of μ^0. Next, suppose that $A \subset B$. There are three cases to consider. Let B be finite. Then A is finite, and $\mu^0(A) = \frac{a}{a+1} < \frac{b}{b+1} = \mu^0(B)$ since $a < b$. Let B be infinite but A be finite. Then $\mu^0(A) = \frac{a}{a+1} < 1 = \mu^0(B)$. Finally, let both A and B be infinite. Then $\mu^0(A) = 1 \leq 1 = \mu^0(B)$.

Now to establish sub-σ-additivity:

$$\mu^0(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^0(A_n).$$

Suppose that at least one of the A_ns is infinite, e.g., A_{n_0}. Then the union $\cup_{n=1}^{\infty} A_n$ is infinite, and hence $\mu^0(\cup_{n=1}^{\infty} A_n) = 1$, whereas $\sum_{n=1}^{\infty} \mu^0(A_n) \geq 1$, since $\mu^0(A_{n_0}) = 1$ and $\mu^0(A_n) \geq 0, n \geq 1$. Next, let all A_n be finite and $\neq \oslash$. Then $\cup_{n=1}^{\infty} A_n$ is infinite, so that $\mu^0(\cup_{n=1}^{\infty} A_n) = 1$. As for the right-hand side, $\mu^0(A_n) = \frac{a_n}{a_n+1} \geq \frac{1}{2}$ for all n, so that $\sum_{n=1}^{\infty} \mu^0(A_n) = \infty$. Finally, suppose that only finitely many of the A_ns are finite, e.g., $A_{n_1}, \ldots, A_{n_k}$. Then, clearly, $\sup(A_{n_1} \cup \ldots \cup A_{n_k}) \leq \sup A_{n_1} + \ldots + \sup A_{n_k}$, so that $\mu^0(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^0(A_n)$. Therefore μ^0 is an outer measure.

(ii) By Remark 6(i), A is μ^0-measurable if

$$\mu^0(D) \geq \mu^0(A \cap D) + \mu^0(A^c \cap D) \text{ for every } D \subseteq \Omega.$$

Also, by Remark 6(ii), $\oslash$ and Ω are μ^0-measurable, so to investigate the last inequality for $\oslash \subset A \subset \Omega$. Consider the following possible cases. Let both A and A^c be infinite, and take $D = \Omega$. Then $\mu^0(\Omega) = 1$, but

$\mu^0(A \cap \Omega) + \mu^0(A^c \cap \Omega) = \mu^0(A) + \mu^0(A^c) = 1 + 1 = 2$, so that the inequality is violated. Let A be infinite but A^c be finite, and take $D = \Omega$. Then $\mu^0(\Omega) = 1$, but $\mu^0(A \cap \Omega) + \mu^0(A^c \cap \Omega) = \mu^0(A) + \mu^0(A^c) = 1 + \frac{c}{c+1}$, $c = \sup A^c$. Again, the inequality is violated. Finally, let A be finite (so that A^c is infinite), and take $D = \Omega$. Once again, $\mu^0(\Omega) = 1$, and $\mu^0(A \cap \Omega) + \mu^0(A^c \cap \Omega) = \mu^0(A) + \mu^0(A^c) = \frac{a}{a+1} + 1$, $a = \sup A$. So, the inequality is violated. The conclusion then is that $\mathcal{A}_0 = \{\oslash, \Omega\}$. #

29. It is immediate since:

$$P(-X \leq -m) = P(X \geq m) \geq \frac{1}{2}, \text{ and}$$
$$P(-X \geq -m) = P(X \leq m) \geq \frac{1}{2}. \#$$

30. By symmetry, we have

$$\begin{aligned} P(X \leq x) &= P(-X \leq x) = P(X \geq -x) \\ &= 1 - P(X < -x) \geq 1 - P(X \leq -x). \end{aligned}$$

For $x = 0$, this becomes

$$P(X \leq 0) \geq 1 - P(x \leq 0), \text{ or } P(x \leq 0) \geq \frac{1}{2}.$$

Again, by symmetry,

$$P(X \geq x) = P(-X \geq x) = P(X \leq -x).$$

For $x = 0$, this relation becomes $P(X \geq 0) = P(X \leq 0)$. But $P(X \leq 0) \geq \frac{1}{2}$ as already shown. Thus, $P(X \geq 0) \geq \frac{1}{2}$, and 0 is a median for X. #

31. From $B \subseteq A \cup B$, we get $\mu^0(B) \leq \mu^0(A \cup B)$. However, $\mu^0(A \cup B) \leq \mu^0(A) + \mu^0(B) = \mu^0(B)$ (by the sub-additivity property of μ^0). Thus, $\mu^0(B) \leq \mu^0(A \cup B) \leq \mu^0(B)$, so that $\mu^0(A \cup B) = \mu^0(B)$. #

32. Let $N = (f \neq g)$, and let $B \in \mathcal{B}$. Then $f^{-1}(B) \in \mathcal{A}$, by assuming that, e.g., f is measurable. Also, $g^{-1}(B) = \{[g^{-1}(B)] \cap N\} \cup \{[g^{-1}(B)] \cap N^c\} = \{[g^{-1}(B)] \cap N\} \cup f^{-1}(B)$ (since $f = g$ on N^c). But $[g^{-1}(B)] \cap N \subseteq N$ with $\mu(N) = 0$. Thus, $[g^{-1}(B)] \cap N$ is in $\mathcal{A}$, and hence $g^{-1}(B)$ is in $\mathcal{A}$. It follows that g is measurable. #

33. Indeed, $B \in \mathcal{B}$, we have $f^{-1}(B) \subseteq A$ with $\mu[f^{-1}(B)] = 0$, so that $f^{-1}(B) \in \mathcal{A}$, and hence f is measurable. #

34. **(i)** We have to show that μ is nonnegative, $\mu(\oslash) = 0$, and μ is σ-additive. Indeed, $\mu(A) = \mu_1(A) + \mu_2(A) \geq 0$; $\mu(\oslash) = \mu_1(\oslash) + \mu_2(\oslash) = 0$; $\mu(\sum_{i=1}^{\infty} A_i) = \mu_1(\sum_{i=1}^{\infty} A_i) + \mu_2(\sum_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_1(A_i) + \sum_{i=1}^{\infty} \mu_2(A_i) = \sum_{i=1}^{\infty}(\mu_1 + \mu_2)(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

(ii) Suppose that, e.g., μ_1 is complete, or more properly, $\mathcal{A}$ is complete with respect to μ_1, which means that $\mathcal{A}$ contains all subsets of the μ_1-null sets. So, let $A \in \mathcal{A}$ with $\mu(A) = 0$. Then $\mu_1(A)(= \mu_2(A)) = 0$. Thus, for an arbitrary $B \subseteq A$, we have $\mu_1(B) \leq \mu_1(A) = 0$ and $B \in \mathcal{A}$. It follows that $\mu(B) \leq \mu(A) = 0$, so that μ is complete. #

35. **(i)** Unions of any two members of $\mathcal{C}_2$ produce elements in $\mathcal{C}_2$ except for two new elements; namely,

$$(A \cap B) \cup (A^c \cap B^c) \quad \text{and} \quad (A \cap B^c) \cup (A^c \cap B).$$

Beyond the obvious results, we have:

$$\begin{aligned}
&A \cup (A^c \cap B) = A \cup B,\ A \cup (A^c \cap B^c) = A \cup B^c;\\
&A^c \cup (A \cap B) = A^c \cup B,\ A^c \cup (A \cap B^c) = A^c \cup B^c;\\
&B \cup (A \cap B^c) = A \cup B,\ B \cup (A^c \cap B^c) = A^c \cup B;\\
&B^c \cup (A \cap B) = A \cup B^c,\ B^c \cup (A^c \cap B) = A^c \cup B^c;\\
&(A \cap B) \cup (A^c \cap B^c) \text{ new element,}\\
&(A \cap B) \cup (A^c \cup B^c) = (A \cap B) \cup (A \cap B)^c = \Omega;\\
&(A \cap B^c) \cup (A^c \cap B) \text{ new element,}\\
&(A \cap B^c) \cup (A^c \cup B) = \Omega;\\
&(A^c \cap B^c) \cup (A \cup B) = (A \cup B)^c \cup (A \cup B) = \Omega.
\end{aligned}$$

(ii) Closeness under complementation is immediate for all elements except, perhaps, for the last two, each of which is the complement of the other. Indeed,

$$\begin{aligned}
[(A \cap B) \cup (A^c \cap B^c)]^c &= (A^c \cup B^c) \cap (A \cup B)\\
&= [(A^c \cup B^c) \cap A] \cup [(A^c \cup B^c) \cap B]\\
&= (A \cap B^c) \cup (A^c \cap B).
\end{aligned}$$

In checking closeness under unions, it suffices to restrict ourselves to forming unions of two elements, one taken from each one of the classes:

$$\begin{aligned}
&\{(A \cap B) \cup (A^c \cap B^c), (A \cap B^c) \cup (A^c \cap B)\},\\
&\{A,\ A^c,\ B,\ B^c,\ A \cap B,\ A \cap B^c,\ A^c \cap B,\ A^c \cap B^c\},
\end{aligned}$$

as well as any two elements from the second class above. To this end, and except for the obvious results, we have:

$$\begin{aligned}
&A \cup [(A \cap B) \cup (A^c \cap B^c)] = A \cup (A^c \cap B^c) = A \cup B^c;\\
&A \cup [(A \cap B^c) \cup (A^c \cap B)] = A \cup (A^c \cap B) = A \cup B;\\
&A^c \cup [(A \cap B) \cup (A^c \cap B^c)] = A^c \cup (A \cap B) = A^c \cup B;\\
&A^c \cup [(A \cap B^c) \cup (A^c \cap B)] = A^c \cup (A \cap B^c) = A^c \cup B^c;\\
&B \cup [(A \cap B) \cup (A^c \cap B^c)] = B \cup (A^c \cap B^c) = A^c \cup B;
\end{aligned}$$

$$B \cup [(A \cap B^c) \cup (A^c \cap B)] = B \cup (A \cap B^c) = A \cup B;$$
$$B^c \cup [(A \cap B) \cup (A^c \cap B^c)] = B^c \cup (A \cap B) = A \cup B^c;$$
$$B^c \cup [(A \cap B^c) \cup (A^c \cap B)] = B^c \cup (A \cap B^c) = B^c;$$
$$(A \cap B) \cup (A^c \cup B^c) = (A \cap B) \cup (A \cap B)^c = \Omega,$$
$$(A \cap B) \cup [(A \cap B^c) \cup (A^c \cap B)] = A \cup (A^c \cap B) = A \cup B;$$
$$(A \cap B^c) \cup (A^c \cup B) = \Omega,$$
$$(A \cap B^c) \cup [(A \cap B) \cup (A^c \cap B^c)] = A \cup (A^c \cap B^c) = A \cup B^c;$$
$$(A^c \cap B) \cup (A \cup B^c) = \Omega,$$
$$(A^c \cap B) \cup [(A \cap B) \cup (A^c \cap B^c)] = A^c \cup (A \cap B) = A^c \cup B;$$
$$(A^c \cap B^c) \cup (A \cup B) = (A \cup B)^c \cup (A \cup B) = \Omega,$$
$$(A^c \cap B^c) \cup [(A \cap B^c) \cup (A^c \cap B)] = B^c \cup (A^c \cap B) = A^c \cup B^c.$$

Again, except for the obvious results, we have:

$$(A \cup B) \cup [(A \cap B) \cup (A^c \cap B^c)] = (A \cup B) \cup (A^c \cap B^c)$$
$$= (A \cup B) \cup (A \cup B)^c = \Omega;$$
$$(A \cup B^c) \cup [(A \cap B^c) \cup (A^c \cap B)] = (A \cup B^c) \cup (A^c \cap B) = \Omega;$$
$$(A^c \cup B) \cup [(A \cap B^c) \cup (A^c \cap B)] = (A^c \cup B) \cup (A \cap B^c) = \Omega;$$
$$(A^c \cup B^c) \cup [(A \cap B) \cup (A^c \cap B^c)] = (A^c \cup B^c) \cup (A \cap B)$$
$$= (A \cup B)^c \cup (A \cap B) = \Omega. \#$$

Chapter 3

Some Modes of Convergence of a Sequence of Random Variables and their Relationships

1. Indeed, $|X_n - X| = (X_n - X)^+ + (X_n - X)^-$, so that $(X_n - X)^+ \leq |X_n - X|$, $(X_n - X)^- \leq |X_n - X|$. Hence, for every $\varepsilon > 0$, $\mu[(X_n - X)^+ \geq \varepsilon] \leq \mu[|X_n - X| \geq \varepsilon] \underset{n\to\infty}{\longrightarrow} 0$, and likewise, $\mu[(X_n - X)^- \geq \varepsilon] \leq \mu[|X_n - X| \geq \varepsilon] \underset{n\to\infty}{\longrightarrow} 0$.
Next, recall that (Exercise 28, Chapter 1) that for any two r.v.s X and Y, $(X + Y)^+ \leq X^+ + Y^+$ and $(X + Y)^- \leq X^- + Y^-$. Hence

$$X_n^+ = ((X_n - X) + X)^+ \leq (X_n - X)^+ + X^+,$$
$$X^+ = ((X - X_n) + X_n)^+ \leq (X - X_n)^+ + X_n^+ = (X_n - X)^- + X_n^+,$$

because, as is easily seen, $(-Z)^+ = Z^-$. Then

$$-(X_n - X)^- \leq X_n^+ - X^+ \leq (X_n - X)^+,$$

or $|X_n^+ - X^+| \leq (X_n - X)^+ + (X_n - X)^- = |X_n - X|$, and therefore

$$\mu(|X_n^+ - X^+| \geq \varepsilon) \leq \mu(|X_n - X| \geq \varepsilon) \underset{n\to\infty}{\longrightarrow} 0,$$

so that $X_n^+ \underset{n\to\infty}{\overset{\mu}{\longrightarrow}} X^+$. Likewise, $X_n^- \underset{n\to\infty}{\overset{\mu}{\longrightarrow}} X^-$. #