Chapter 2
Spatial Descriptions and Transformations

Exercises

2.1) $R = \text{rot}(\hat{x}, \phi)\,\text{rot}(\hat{z}, \theta)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & -S\theta & 0 \\ C\phi S\theta & C\phi C\theta & -S\phi \\ S\phi S\theta & S\phi C\theta & C\phi \end{bmatrix}$$

2.2) $R = \text{rot}(\hat{x}, 45^\circ)\,\text{rot}(\hat{y}, 30^\circ)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .707 & -.707 \\ 0 & .707 & .707 \end{bmatrix} \begin{bmatrix} .866 & 0 & .5 \\ 0 & 1 & 0 \\ -.5 & 0 & .866 \end{bmatrix}$$

$$= \begin{bmatrix} .866 & 0 & .5 \\ .353 & .707 & -.612 \\ -.353 & .707 & .612 \end{bmatrix}$$

2.3) Since rotations are performed about axes of the frame being rotated, these are Euler-Angle style rotations:

$R = \text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi)$

We might also use the following reasoning:

$${}^{A}_{B}R(\theta, \phi) = {}^{B}_{A}R^{-1}(\theta, \phi)$$

$$= [\text{rot}(\hat{x}, -\phi)\,\text{rot}(\hat{z}, -\theta)]^{-1}$$

$$= \text{rot}^{-1}(\hat{z}, -\theta)\,\text{rot}^{-1}(\hat{x}, -\phi)$$

$$= \text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi)$$

Yet another way of viewing the same operation:

1st rotate by $\text{rot}(\hat{z}, \theta)$

2nd rotate by $\text{rot}(\hat{z}, \theta)\,\text{rot}(\hat{x}, \phi)\,\text{rot}^{-1}(z, \theta)$

2.3) *(continued)*

(This is a similarity transform)

Composing these two rotations:

$$= \mathrm{rot}(\hat{z}, \theta)\, \mathrm{rot}(\hat{x}, \phi)\, \mathrm{rot}^{-1}(z, \theta) \cdot \mathrm{rot}(\hat{z}, \theta)$$

$$= \mathrm{rot}(\hat{z}, \theta)\, \mathrm{rot}(\hat{x}, \phi)$$

$$= \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & -S\theta C\phi & S\theta S\phi \\ S\theta & C\theta C\phi & -C\theta S\phi \\ 0 & S\phi & C\phi \end{bmatrix}$$

2.4) This is the same as 2.3 only with numbers.

$$R = \mathrm{rot}(\hat{z}, 30^\circ)\, \mathrm{rot}(\hat{x}, 45)$$

$$= \begin{bmatrix} .866 & -.353 & .353 \\ .50 & .612 & -.612 \\ 0 & .707 & .707 \end{bmatrix}$$

2.5) If V_i is an eigenvector of R, then

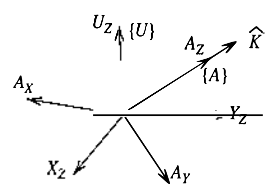
$$RV_i = 7V_i$$

If the eigenvalue associated with V_i is 1, then

$$RV_i = V_i$$

Hence the vector is not changed by the rotation R.
So V_i is the axis of rotation.

2.6) Imagine a frame $\{A\}$ whose $\hat{z}$ axis is aligned with the direction $\hat{k}$:



Then, the rotation with rotates vectors about $\hat{k}$ by θ degrees could be written:

$$R = {}^U_A R\, \mathrm{rot}({}^A\hat{z}, \theta)\, {}^A_U R \quad [1]$$

We write the description of $\{A\}$ in $\{U\}$ as:

$${}^U_A R = \begin{bmatrix} A & D & K_x \\ B & E & K_y \\ C & F & K_z \end{bmatrix}$$

If we multiply out Eq. [1] above, and then simplify using $A^2 + B^2 + C^2 = 1$, $D^2 + E^2 + F^2 = 1$, $[ABC] \cdot [DEF] = 0$, $[ABC] \otimes [DEF] = [K_x K_y K_z]$ we arrive at Eq. (2.80) in the book. Also, see [R. Paul]* page 25.

2.7) Let $R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$

(1) Compute $R_{11} + R_{22} + R_{33} = N$

(2) If $N = 3$, then $\theta = A\cos\left(\frac{N-1}{2}\right) = 0°$. Since rotation is zero, $\hat{K}$ is arbitrary.

(3) If $N = -1$, then $\theta = A\cos(-1) = 180°$. In this case Eq. (2.80) becomes:

$$\text{rot}(\hat{K}, 180°) = \begin{bmatrix} 2K_x^2 - 1 & 2K_xK_y & 2K_xK_z \\ 2K_xK_y & 2K_y^2 - 1 & 2K_yK_z \\ 2K_xK_z & 2K_yK_z & 2K_z^2 - 1 \end{bmatrix}$$

so:

$$2K_x^2 - 1 = R_{11} \Rightarrow K_x = \pm\sqrt{(R_{11}+1)/2}$$

$$2K_xK_y = R_{21} \Rightarrow K_y = R_{12}/2K_x$$

$$2K_xK_z = R_{31} \Rightarrow K_z = R_{31}/2K_x$$

However, if $K_x \cong 0$, then this is ill-defined, so use a different column for solution (not the first column as above).

(4) If $-1 < N < 3$ (so that $0 < \theta < 180°$) the use Eq. (2.82) in book.

2.8) Procedure RMTOAA is given essentially in the solution to 2.7. However, writing clean code to check the various cases is a good exercise in itself. Procedure AATORM is given by Eq. (2.80) and is easy.

2.9) The subroutines encode Eq. (2.64) and equations (2.66), (2.67), and (2.68).

2.10) The subroutines follow from eq. (2.72) and equations (2.74), (2.75), and (2.76).

2.11) When they represent rotations about the same axis.

2.12) Velocity is a "free vector" and only will be affected by rotation, and not by translation:

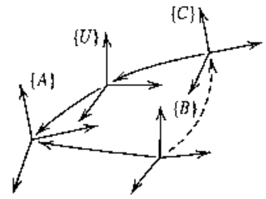
$${}^AV = {}^A_BR\,{}^BV = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$${}^AV = [-1.34 \quad 22.32 \quad 30.0]^T$$

2.13) By just following arrows, and reversing (by inversion) where needed, we have:

$${}^B_C T = {}^B_A T \; {}^U_A T^{-1} \; {}^C_U T^{-1}$$

Inverting a transform is done using eq. (2.40) in book. Rest is boring.



2.14) This rotation can be written as:

$${}^A_B T = \text{trans}({}^A\hat{P}, |{}^A P|)\,\text{rot}(\hat{K}, \theta)\text{trans}(-{}^A\hat{P}, |{}^A P|)$$

Where $\text{rot}(\hat{K}, \theta)$ is written as in eq. (2.77),

$$\text{And} \quad \text{trans}({}^A\hat{P}, |{}^A P|) = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{And} \quad \text{trans}(-{}^A\hat{P}, |{}^A P|) = \begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & -P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying out we get:

$${}^A_B T = \left[\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & Q_x \\ R_{21} & R_{22} & R_{23} & Q_y \\ R_{31} & R_{32} & R_{33} & Q_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where the R_{ij} are given be eq. (2.77). And:

$$Q_x = P_x - P_x(K_x^2 V\theta + C\theta) - P_y(K_x K_y V\theta - K_z S\theta)$$
$$- P_z(K_x K_z V\theta + K_y S\theta)$$

$$Q_y = P_y - P_x(K_x K_y V\theta + K_z S\theta) - P_y(K_y^2 V\theta + C\theta)$$
$$- P_z(K_y K_z V\theta + K_x S\theta)$$

$$Q_z = P_z - P_x(K_x K_z V\theta - K_y S\theta) - P_y(K_y K_z V\theta + K_x S\theta)$$
$$- P_z(K_z^2 V\theta + C\theta)$$

2.15) Recall that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

so $e^{K\theta} = I + K\theta + \frac{1}{2!}K^2\theta^2 + \frac{1}{3!}K^3\theta^3 + \cdots$

$$K^2 = \begin{bmatrix} -K_y^2 - K_z^2 & K_x K_y & K_x K_z \\ K_x K_y & -K_x^2 - K_z^2 & K_y K_z \\ K_x K_z & K_y K_z & -K_x^2 - K_y^2 \end{bmatrix}$$

Writing out the (1,2) element of $e^{K\theta}$ (as an example) we have:

$$(e^{K\theta})_{1,2} = O + (-K_z)\theta + \frac{1}{2!}(K_x K_y)\theta^2 + \frac{1}{3!}K_2\theta^3 + \cdots$$

Recall that:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots.$$

$$V\theta = 1 - C\theta = \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots$$

We can write:

$$(e^{K\theta})_{1,2} = (-K_z)\theta + \frac{1}{3!}K_2\theta^3 - \frac{1}{5!}K_2\theta^5 + \cdots$$

$$+ \frac{1}{2!}(K_x K_y)\theta^2 - \frac{1}{4}(K_x K_y)\theta^4 + \cdots$$

Or:

$$(e^{K\theta})_{1,2} = -K_2 S\theta + K_x K_y V\theta$$

Which is the value given in (2.80). Other elements may be checked similarly.

2.16) In method 1, the multiplication of two 3×3 matrices requires 27 multiplications and 18 additions. the computation of ${}^A_BR$ takes two matrix multiplies, or 57 multiplications, 36 additions the computation of ${}^A_BR\ {}^BP$ is 9 mult. and 6 additions. Hence, in one second this method will require:

$30 \times$ computation of ${}^A_DR = 30 \times 54$ mult.

30×36 add.

$100 \times$ computation of ${}^Ap = 100 \times 9$ mult.

100×6 add.

Total $= 2520$ mult., 1680 Add.

In method two, computation of ${}^C_DR\ {}^DP$ requires 9 mult. and 6 add.; likewise the computation of ${}^B_CR\ {}^CP$ and ${}^A_BR\ {}^BP$, for a total of 27 mult. and 18 add. These must occur 100 times/sec., so in one second we have:

27×100 mult. $= 2700$ mult.

18×100 add. $= 1800$ add.

Therefore, method 1 is superior, but not by much.

2.17) $${}^AP = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} R\cos\theta \\ R\sin\theta \\ Z \end{bmatrix}$$

2.18) $${}^AP = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} R\cos\alpha\cos\beta \\ R\sin\alpha\cos\beta \\ R\sin\beta \end{bmatrix}$$

2.19) In the z-y-z Euler Angle set, the first rotation is:

$R_1 = \mathrm{rot}(\hat{z}, \alpha)$

The second rotation expressed in fixed coordinates is:

$R_2 = \mathrm{rot}(\hat{z}, \alpha)\,\mathrm{rot}(\hat{y}, \beta)\,\mathrm{rot}^{-1}(\hat{z}, \alpha)$

The third is:

$R_3 = (R_2R_1)\,\mathrm{rot}(\hat{z}, \gamma)(R_2R_1)^{-1}$

The result is:

$R = R_3R_2R_1 = \mathrm{rot}(\hat{z}, \alpha)\,\mathrm{rot}(\hat{y}, \beta)\,\mathrm{rot}(\hat{z}, \gamma)$

Which can be multiplied out to give the result of (2.72).

2.20) This is easily derived if you work backwards. i.e, substitute into Rodriquez's formula wherever $\hat{K} \otimes \hat{Q}$ or $\hat{K} \cdot \hat{Q}$ occur, collect terms, and you'll get (2.80).

2.21) Just use the given approximations in (2.80) to obtain:

$$R_K(\delta\theta) = \begin{bmatrix} 1 & -K_Z\delta\theta & K_Y\delta\theta \\ K_Z\delta\theta & 1 & -K_X\delta\theta \\ -K_Y\delta\theta & K_X\delta\theta & 1 \end{bmatrix}$$

More on this in Chapter 5.

2.22) So, given $R_1 = R_J(\alpha)$ and $R_2 = R_K(\beta)$ with $\alpha \ll 1$ and $\beta \ll 1$; show $R_1R_2 = R_2R_1$ if we form the product R_1R_2 and use $\alpha\beta \cong 0$ we have:

$$R_1R_2 = \begin{bmatrix} 1 & -J_Z\alpha - K_Z\beta & J_Y\alpha + K_Y\beta \\ J_Z\alpha + K_Z\beta & 1 & -J_X\alpha - K_X\beta \\ -J_Y\alpha - K_Y\beta & J_X\alpha + K_X\beta & 1 \end{bmatrix}$$

We see that j and k, as well as α and β appear symmetrically, so $R_1R_2 = R_2R_1$.

2.23) By definition ${}^U P_{\text{AORG}} = {}^U P_1$. Next, the vector from P_1 TO P_2 is a vector along the positive x-axis, so: ${}^U X_A = {}^U P_2 - {}^U P_1$, which normalized is: ${}^U \hat{X}_A = {}^U X_A / \|{}^U X_A\|$.

Now ${}^U V = {}^U P_3 - {}^U P_1$. A vector parallel to the positive Y_A-axis, can be formed using the Gram-Schmidt orthogonalization:

$${}^U Y_A = {}^U V - ({}^U V \cdot {}^U \hat{X}_A)\, {}^U \hat{X}_A$$

and

$${}^U \hat{Y}_A = {}^U Y_A / |{}^U Y_A|$$

Finally, the unit vector ${}^U \hat{Z}_A$ can be found by a simple cross-product:

$${}^U \hat{Z}_A = {}^U \hat{X}_A \otimes {}^U \hat{Y}_A$$

Now:

$${}^U_A T = \left[\begin{array}{ccc|c} {}^U \hat{X}_A & {}^U \hat{Y}_A & {}^U \hat{Z}_A & {}^U P_{\text{Aorb}} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

2.24) This is a bit tricky here's most of it: ${}^{A}P = {}^{A}_{B}R\,{}^{B}P$ doesn't change length, so

$$\therefore\ {}^{A}P \cdot {}^{A}P - {}^{B}P \cdot {}^{B}P = 0$$

$$\underbrace{({}^{A}P - {}^{B}P)}_{F} \cdot \underbrace{({}^{A}P + {}^{B}P)}_{G} = 0$$

$$\therefore\ F \perp G$$

$$F = ({}^{A}_{B}R - I)\,{}^{B}P \quad G = ({}^{A}_{B}R + I)\ {}^{B}P$$

${}^{B}P = ({}^{A}_{B}R + I)^{-1}G$ (You can show ${}^{A}_{B}R + I$ non-shawar)

$$\therefore\ F = \underbrace{({}^{A}_{B}R - I)({}^{A}_{B}R + I)^{-1}}_{B}\,G$$

so, $F = BG$

(Now, one can show that if $X \cdot Y = 0$ and if $X = AY$ then a is skew-symmetric)

$\therefore\ B \in$ skew-sym matrices

$$({}^{A}_{B}R - I) = B({}^{A}_{B}R + I)$$

$(I - B){}^{A}_{B}R = I + B$ (now can show (I-B) non-show)

$\therefore$ $\boxed{{}^{A}_{B}R = (I - B)^{-1}(I + B)}$ Q.E.D.

2.25) Def. of e-val: $\lambda_i V_i = RV_i$ from physical insight we know $\underline{V_i = RV_i}$ when V_i is the axis of rotation, $\therefore$ $\boxed{\lambda_1 = 1}$ from (2.80) one can show (with some work) that det $(R) = 1$. From linear algebra we have:

$$\Sigma_i \lambda_i = \text{trace(r)} \quad \pi_i \lambda_i = \text{det(r)}$$

$\therefore\ \lambda_1\lambda_2\lambda_3 = 1$ or $\underline{\lambda_2\lambda_3 = 1}$ **[1]**

And compute the trace(r) from (2.80) to get

$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2\cos\theta$ or $\underline{\lambda_2 + \lambda_3 = 2\cos\theta}$ **[2]**

Now solve [1] & [2] above for λ_2 and λ_3 to get

$$\lambda_2 + \frac{1}{\lambda_2} = 2\cos\theta; \quad \lambda_2^2 - 2\cos\theta\lambda_2 + 1 = 0$$

$$\lambda_2 = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$\lambda_2 = \cos\theta \pm i\sin\theta$$

$$\therefore\ \lambda_1 = 1$$

$$\lambda_2 = \cos\theta + i\sin\theta = e^{i\theta}$$

$\lambda_3 = \cos\theta - i\sin\theta = e^{-i\theta}$ Q.E.D.

2.26) Somebody please send me a simple proof of this.
(For any given Euler convention it is not hard.)

2.27) ${}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.28) ${}^{A}_{C}T = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.29) ${}^{B}_{C}T = \begin{bmatrix} 0 & 0.5 & -0.866 & 0 \\ 0 & -0.866 & -0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.30) ${}^{C}_{A}T = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -0.5 & 0.866 & 0 & 3 * 0.5 \\ 0.866 & -0.5 & 0 & -3 * 0.866 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.31) ${}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.32) ${}^{A}_{C}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -3 \\ 0.5 & -0.866 & 0 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.33) ${}^{B}_{C}T = \begin{bmatrix} -0.866 & -0.5 & 0 & 3 \\ 0 & 0 & +1 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.34) ${}^{C}_{A}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -3^{*}.86 + 2 \\ 0.5 & -0.866 & 0 & -4^{*}.86 - 1.5 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2.35) Any R can be given: $R = R_X(\alpha)R_Y(\beta)R_Z(\gamma)$ and $\det(R) = \det(R_X(\alpha)) \cdot \det(R_Y(\beta)) \cdot \det(R_Z(\gamma))$

Using the formulas for rotation about a principle axis ((2.77) – (2.79)) its easy to show

$\det(R_x(\alpha)) = 1 \quad \forall\alpha$

$\det(R_y(\beta)) = 1 \quad \forall\beta$

$\det(R_z(\gamma)) = 1 \quad \forall\gamma$

$\therefore \ \det(R) = 1 \cdot 1 \cdot 1 = 1$ Q.E.D.

2.36) From Cayley's formula, the number of parameters needed to specify a rotation is the number of free parameters in an NXN skew symmetric matrix, which is $\frac{1}{2}(N^2 - N)$. The translational degrees of freedom are, of course n, so total is:

$\mathrm{dof}(N) = N + \frac{1}{2}(N^2 - N) = \frac{1}{2}(N^2 + N)$ Q.E.D.

2.37) Form (2,4) element of $-{}^A_BR^{T\,A}P_{\mathrm{borb}}$

To get: -6.4

2.38) $V_1^T V_2 = \cos\theta$, R preserves angles, so,

$(RV_1)^T(RV_2) = V_1^T V_2$

$V_1^T R^T R V_2 = V_1^T V_2 \quad \therefore \quad R^T R = I \Rightarrow R^T = R^{-1}$

2.39) $\varepsilon_y^2 = \frac{1}{y}(1 + r_{11} + r_{22} + r_{33})$

$\varepsilon_y^2 >$ epsilon?

<table>
<tr><th>Yes</th><th>No</th></tr>
<tr><td>$\varepsilon_y = \sqrt{\varepsilon_y^2}$
$\varepsilon_1 = (r_{23} - r_{32})/4\varepsilon_y$
$\varepsilon_2 = (r_{31} - r_{13})/4\varepsilon_y$
$\varepsilon_3 = (r_{12} - r_{21})/4\varepsilon_y$</td><td>$\varepsilon_y = 0$
$\varepsilon_1^2 = -1/2(r_{22} + r_{33})$
$\varepsilon_1^2 >$ epsilon?<table>
<tr><th>Yes</th><th>No</th></tr>
<tr><td>$\varepsilon_1 = \sqrt{\varepsilon_1^2}$
$\varepsilon_2 = r_{12}/2\varepsilon_1$
$\varepsilon_3 = r_{13}/2\varepsilon_1$</td><td>$\varepsilon_1 = 0$
$\varepsilon_2^2 = \frac{1}{2}(1 - r_{33})$
$\varepsilon_2^2 >$ epsilon?<table>
<tr><th>Yes</th><th>No</th></tr>
<tr><td>$\varepsilon_2 = \sqrt{\varepsilon_2^2}$
$\varepsilon_3 = r_{23}/2\varepsilon_2$</td><td>$\varepsilon_2 = 0$
$\varepsilon_3 = 1$</td></tr>
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2.40) Similar to algorithms of Section 2.8.

2.41) Similar to algorithms of Section 2.8.

Chapter 2 Solutions for Introduction to Robotics

1. a) Use (2.3) to obtain

$$ {}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} $$

b) Use (2.74) to get

$$ \begin{aligned} \alpha &= 90 \text{ degrees} \\ \beta &= 90 \text{ degrees} \\ \gamma &= -90 \text{ degrees} \end{aligned} $$

2. a) Use (2.64) to obtain

$$ {}^{A}_{B}R = \begin{bmatrix} .330 & -.770 & .547 \\ .908 & .418 & .0396 \\ -.259 & .483 & .837 \end{bmatrix} $$

b) Answer is the same as in (a) according to (2.71)

3. Use (2.19) to obtain the transformation matrices. The rotation is X-Y-Z fixed angles, so use (2.64) for that 3×3 submatrix, with angles

$$ \begin{aligned} \gamma &= 0 \text{ degrees} \\ \beta &= -\sin^{-1}\left(\frac{\text{tripod\_height}}{\text{distance\_along\_optical\_axis}}\right) = -\sin^{-1}\left(\frac{1.5}{5}\right) = -107 \text{ degrees} \\ \alpha_C &= 0 \text{ degrees} \\ \alpha_D &= 120 \text{ degrees} \\ \alpha_E &= 240 \text{ degrees} \end{aligned} $$

The position vectors to the camera-frame origins are

$$
{}^{B}P_{CORG} = \begin{bmatrix} \text{horizontal\_distance} \\ 0 \\ \text{tripod\_height} \end{bmatrix} = \begin{bmatrix} 4.77 \\ 0 \\ 1.50 \end{bmatrix}
$$

$$
{}^{B}P_{DORG} = \begin{bmatrix} \text{horizontal\_distance} \times \cos\alpha_D \\ \text{horizontal\_distance} \times \sin\alpha_D \\ \text{tripod\_height} \end{bmatrix} = \begin{bmatrix} -2.39 \\ 4.13 \\ 1.5 \end{bmatrix}
$$

$$
{}^{B}P_{EORG} = \begin{bmatrix} \text{horizontal\_distance} \times \cos\alpha_E \\ \text{horizontal\_distance} \times \sin\alpha_E \\ \text{tripod\_height} \end{bmatrix} = \begin{bmatrix} -2.38 \\ -4.13 \\ 1.50 \end{bmatrix},
$$

where $\text{horizontal\_distance} = \sqrt{(\text{distance\_along\_optical\_axis})^2 - (\text{tripod\_height})^2}$. Combining the rotation and translation yields the transformation matrices via (2.19) as

$$
{}^{B}_{C}T = \begin{bmatrix} -.300 & 0 & -.954 & 4.77 \\ 0 & 1.00 & 0 & 0 \\ .954 & 0 & -.300 & 1.50 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{D}T = \begin{bmatrix} .150 & -.866 & .477 & -2.39 \\ -.260 & -.500 & -.826 & 4.13 \\ .954 & 0 & -.300 & 1.50 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{E}T = \begin{bmatrix} .150 & .866 & .477 & -2.39 \\ .260 & -.500 & .826 & -4.13 \\ .954 & 0 & -.300 & 1.50 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

4. The camera-frame origin is located at ${}^{B}P_{CORG} = [7 \quad -2 \quad 5]^{\mathsf{T}}$. Use (2.19) to get the transformation, ${}^{B}_{C}T$. The rotation is Z-Y-X Euler angles, so use (2.71) with

$$
\begin{aligned}
\alpha &= 0 \text{ degrees} \\
\beta &= -110 \text{ degrees} \\
\gamma &= -20 \text{ degrees}
\end{aligned}
$$

to get

$$
{}^{B}_{C}T = \begin{bmatrix} -.342 & .321 & -.883 & 7.00 \\ 0 & .940 & .342 & -2.00 \\ .940 & .117 & -.321 & 5.00 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

5. Let

$$^{B}P_1 = \; ^{B}P_0 + 5\,^{B}V_0 = [9.5 \quad 1.00 \quad -1.50]^{\mathsf{T}}$$

The object's position in $\{A\}$ is

$$^{A}P_1 = \, ^{A}_{B}T\,^{B}P_1 = [-4.89 \quad 2.11 \quad 3.60]^{\mathsf{T}}$$

6. *(2.1)*

$$\begin{aligned} R &= \mathrm{rot}(\hat{Y},\, \phi)\,\mathrm{rot}(\hat{Z},\, \theta) \\ &= \begin{bmatrix} \mathrm{c}\phi & 0 & \mathrm{s}\phi \\ 0 & 1 & 0 \\ -\mathrm{s}\phi & 0 & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta & -\mathrm{s}\theta & 0 \\ \mathrm{s}\theta & \mathrm{c}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{c}\phi\,\mathrm{c}\theta & -\mathrm{c}\phi\,\mathrm{s}\theta & \mathrm{s}\phi \\ \mathrm{s}\theta & \mathrm{c}\theta & 0 \\ -\mathrm{s}\phi\,\mathrm{c}\theta & \mathrm{s}\phi\,\mathrm{s}\theta & \mathrm{c}\phi \end{bmatrix} \end{aligned}$$

7. *(2.2)*

$$\begin{aligned} R &= \mathrm{rot}(\hat{X},\, 60)\,\mathrm{rot}(\hat{Y},\, -45) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .500 & -.866 \\ 0 & .866 & .500 \end{bmatrix} \begin{bmatrix} .707 & 0 & -.707 \\ 0 & 1 & 0 \\ .707 & 0 & .707 \end{bmatrix} \\ &= \begin{bmatrix} .707 & 0 & -.707 \\ -.612 & .500 & -.612 \\ .353 & .866 & .353 \end{bmatrix} \end{aligned}$$

8. *(2.12)* Velocity is a "free vector" and only will be affected by rotation, and not by translation:

$$\begin{aligned} ^{A}V = \, ^{A}_{B}R\,^{B}V &= \begin{bmatrix} .707 & 0 & -.707 \\ -.612 & .500 & -.612 \\ .353 & .866 & .353 \end{bmatrix} \begin{bmatrix} 30.0 \\ 40.0 \\ 50.0 \end{bmatrix} \\ &= [-14.1 \quad -29.0 \quad 62.9]^{\mathsf{T}} \end{aligned}$$

9. *(2.31)*

$$^{C}_{B}T = \begin{bmatrix} 0 & 0 & -1 & 2 \\ .500 & -.866 & 0 & 0 \\ -.866 & -.500 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10. *(2.37)* Using (2.45) we get that

$$^{B}P_{AORG} = -\,^{A}_{B}R^{\mathsf{T}}\,^{A}P_{AORG} = -\begin{bmatrix} .25 & .87 & .43 \\ .43 & -.50 & .75 \\ .86 & .00 & -.50 \end{bmatrix} \begin{bmatrix} 5.0 \\ -4.0 \\ 3.0 \end{bmatrix} = \begin{bmatrix} .94 \\ -6.4 \\ -2.8 \end{bmatrix}$$